

Consensus versus persistence of disagreement in opinion formation: the role of zealots

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We consider a general class of three-state models where individuals hold one of two opposite opinions, or are neutral, and exchange opinions in generic pairwise interactions. We show that when opinions spread in a population where a fraction of individuals (zealots) maintain unshakably their opinion, one of four qualitatively distinct kinds of collective dynamics arise, depending on the specific rules governing the social interactions. Unsurprisingly, when their density is high, zealots drive the whole population to consensus on their opinion. For low densities a rich phase diagram emerges: a finite population of dissenters can survive and be the only stationary state or may need a critical mass or dissenters to be sustained; the critical mass may vanish or not as the density is reduced; the transition to the high density regime can be smooth or abrupt, and shows interesting hysteretic effects. For each choice of the interaction rules we calculate the critical density of zealots above which diverse opinions cannot survive.

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I. INTRODUCTION

Consensus often develops spontaneously in societies, leading to the adoption of widespread opinions, ideologies, traditions. On the other hand disagreement among individuals is also very common, even on major issues. Disagreement is not always due to lack of communication, and in some cases it persists even when individuals are allowed to exchange opinions and debate for long time. What are the mechanisms that lead to consensus, and which ingredients cause instead persistent disagreement? Although broadcasting media such as television and advertisement have a significant impact on individuals making their opinion, macroscopic opinion shifts in social groups may be caused by peer-to-peer interactions occurring within the social network. People form and reconsider their opinion while constantly interacting with others, and are often exposed to a considerable pressure to conform to the opinion of their friends and neighbors [1, 2]. Yet, on certain matters, some individuals are unshakable in their opinion, and therefore totally insensitive to peer pressure. The aim of this paper is to analyse the role of such “stubborn” individuals, completely reluctant to change, in determining the onset of consensus or rather the persistence of disagreement.

The effect of stubborn individuals (denoted also as committed, zealots or inflexibles) on opinion dynamics has attracted considerable interest in the past years. Early investigations analyzed the effect of a single [3] or multiple individuals [4, 5] in the context of the classical voter model on lattices. The presence of zealots has then been studied in many other contexts, including different types of dynamics, various interaction patterns, the presence of single, multiple or competing zealots [6–14]. Among the most interesting and general results is the ob-

servation [7] of a transition occurring as the fraction of zealots in the system exceeds some threshold: When the density of zealots is above threshold the system is driven to consensus; otherwise, alternative opinions survive in the population. Previous studies have generally focused on models which, in the absence of zealots, exhibit a symmetric dynamics, i.e., no opinion is a priori favored by the spontaneous evolution. Here we consider instead a very general class of 3-state models where the dynamics may either favor or disfavor the opinion held by the zealots. In this wider context new nontrivial phenomena occur, including the counterintuitive possibility that an opinion disfavored by the dynamics and opposed by zealots survives and is even adopted by the majority of individuals. In a previous paper [15] we studied a similar general three-state opinion dynamics model focusing on the effect of media. The effect of media was schematized in the simplest possible way by assuming that people conform to the media recommendation at a constant rate and independently on their current state. The presence of inflexible individuals considered here can also be interpreted, at the mean-field level, as an effective external field, but more complex than the one considered in [15]: it induces spontaneous transitions either to the opinion supported by the media or to the neutral state, and the transition rates depend on the current state. The model presented here includes that of Ref. [13] as a special case.

The paper is structured as follows. After the description of the general model in Section II, we write down the mean-field equation for the dynamics in Section III. The position and stability of their stationary solutions, depending on the model parameters, and the consequent different shapes of the phase diagram are illustrated in Section IV, while the detailed derivation of the results is deferred to Section V. Section VI contains a discussion

		A, A	B, B	U, U	A, U	B, U	A, B
$A - A$		1	0	0	0	0	0
$B - B$		0	1	0	0	0	0
$U - U$		0	0	1	0	0	0
$A - U$		φ_2	0	0	$1 - \varphi_2$	0	0
$B - U$		0	γ_2	0	0	$1 - \gamma_2$	0
$A - B$	$A \notin Z$	α_1	α_2	α_3	α_4	α_5	α_6
	$A \in Z$	β_1	0	0	β_4	0	β_6

TABLE I. Each row in the table corresponds to an interaction, and each column to a possible outcome. Elements in the table indicate the probabilities of each possible outcome for the given interaction. The last two rows differ because the outcome of the $A - B$ interaction depends on whether the agent in state A is committed or not. The coefficients α and β are normalized: $\sum_{i=1}^6 \alpha_i = 1$, $\beta_1 + \beta_4 + \beta_6 = 1$. Note that the Z sub-population is constant.

of the results and concluding remarks.

II. THE MODEL

We analyze a model of pairwise social influence for opinion dynamics in a society consisting of two types of agents: regular agents, who update their beliefs according to the information that they receive from their social neighbors; and stubborn agents, who never update their opinion. Stubborn agents might for example represent political activists or customers involved in the marketing process for a crowdsourced campaign. We include agents with no preferred opinion: neutral agents, who are uninformed or undecided. This allows significant change from models where only two opinions are allowed. Each agent can be in one of three states: holding opinion A , holding an opposing opinion B , or being undecided (U). A finite fraction p of the population forms a sub-population Z of agents committed to the opinion A : they never change state. We indicate with n_A , n_B , n_U the fraction of uncommitted agents in the A , B , and U state, respectively, so that $n_A + n_B + n_U + p = 1$. The interactions rules are specified in Table I. Interactions involving two individuals in the same state ($A - A$, committed or not, $B - B$, $U - U$) have no effect. In $A - U$ ($B - U$) interactions undecided agents may adopt their partner's opinion with given probability: an agent holding opinion A (B) has a probability φ_2 (γ_2) to convince the U agent, while undecided agents do not alter the opinion of A (B). We assume in general $\varphi_2 \neq \gamma_2$, allowing for A and B to have unequal persuasiveness. The outcome of interactions among agents holding opposite opinions ($A - B$) depends on whether the agent holding opinion A is committed or not. When A is uncommitted ($A \notin Z$), the interaction may have any outcome: each of the two agents may either keep her opinion, change it to match the opinion of the partner, or get confused and turn to the undecided state, in any combination. We indicate with α_i , $\{i = 1, 6\}$ ($\sum_{i=1}^6 \alpha_i = 1$) the probabilities of

the six possible outcomes (see Table I). Interactions between an agent committed to opinion A ($A \in Z$) and an agent holding opinion B have instead only three possible outcomes, corresponding to the three possible states of B after the interaction, since the state of A is unaltered. We indicate with $\{\beta_i, i = 1, 4, 6\}$ ($\beta_1 + \beta_4 + \beta_6 = 1$) the probabilities of these outcomes (see Table I).

III. THE DYNAMICS ON A COMPLETE GRAPH

We now write the equations for the dynamics of this general model on a complete graph of infinite size (i.e. in mean-field), describing the time evolution of the density of uncommitted agents in each state:

$$\begin{cases} \dot{n}_A = 2\beta_1 p n_B + 2\varphi_1 n_A n_B + 2\varphi_2 (n_A + p) n_U \\ \dot{n}_B = -2(\beta_1 + \beta_4) p n_B + 2\gamma_1 n_A n_B + 2\gamma_2 n_B n_U \end{cases} \quad (1)$$

where $\varphi_1 = \alpha_1 - \alpha_2 - \alpha_3 - \alpha_5$ and $\gamma_1 = -\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4$ represent the net gain in A (B) states in an $A - B$ interaction. The equation for \dot{n}_U can be derived from the normalization condition. Defining $r = p/(1 - p)$, and $\tilde{n}_i = n_i/(1 - p)$ ($i = A, B, U$) as the fraction of uncommitted agents respectively in the A , B and U state, normalized with respect to the total uncommitted population ($\tilde{n}_A + \tilde{n}_B + \tilde{n}_U = 1$), Eqs. (1) can be rewritten as:

$$\begin{cases} \dot{\tilde{n}}_A(1 + r) = 2r\beta_1 \tilde{n}_B + 2r\varphi_2 \tilde{n}_U + 2\varphi_1 \tilde{n}_A \tilde{n}_B + 2\varphi_2 \tilde{n}_A \tilde{n}_U \\ \dot{\tilde{n}}_B(1 + r) = -2r(\beta_1 + \beta_4) \tilde{n}_B + 2\gamma_1 \tilde{n}_A \tilde{n}_B + 2\gamma_2 \tilde{n}_B \tilde{n}_U \end{cases} \quad (2)$$

From Eqs. (2) it is clear that, in terms of the the densities of uncommitted sub-populations and with a proper rescaling of the time variable, the system maps exactly onto a system of uncommitted agents in an external field acting as an exogenous one-to-many broadcasting source. Interactions and the effect of the external field are specified as follows: The external field induces individuals holding opinion B to switch to opinion A at rate $r\beta_1$, and to become undecided (U) at rate $r\beta_4$; undecided individuals acquire opinion A at rate $r\varphi_2$. The interactions are those described in Table I for uncommitted agents. A similar case was studied in [15] where the focus was on the interplay between external media and interpersonal influence events occurring within the social network. In that case the effect of media was schematized in the simplest possible way by assuming that people conform to the media recommendation at a constant rate and independently on their current state. The model presented here can be seen as a generalization of Ref. [15]. The presence of zealots introduces in the dynamical equations additional terms, that can be interpreted as external fields inducing the spontaneous transitions $B \rightarrow A$ and $U \rightarrow A$ (but at *different* rates), as well as the transition $B \rightarrow U$. The model studied in Ref. [15] is recovered as a special case of the model studied here, with $\beta_1 = \varphi_2$, $\beta_4 = 0$, and $r \rightarrow r/2\varphi_2$, so that $r_{B \rightarrow A} = r$, $r_{B \rightarrow U} = 0$, $r_{U \rightarrow A} = 2\varphi_2 p = r$. For $r = 0$ the two models coincide.

IV. COLLECTIVE BEHAVIOR AT STATIONARITY

Choosing $x = \tilde{n}_B$ and $y = \tilde{n}_A$ as independent variables, the stationarity condition in Eq. (2) reads:

$$\begin{cases} y^2 - y(1 - r + \varphi x) + r\alpha x - r = 0, & \mathcal{C}_1 \\ x(\gamma y - x + 1 - r\beta) = 0 & \mathcal{C}_2, \end{cases} \quad (3)$$

where $\varphi = (\varphi_1 - \varphi_2)/\varphi_2$, $\gamma = (\gamma_1 - \gamma_2)/\gamma_2$, $\alpha = \varphi_2 - \beta_1/\varphi_2$, and $\beta = (\beta_1 + \beta_4)/\gamma_2$, and we assume $\varphi_2 \neq 0$, $\gamma_2 \neq 0$. The stationary solutions of the dynamic equations are therefore given by the intersections of the two conic sections \mathcal{C}_1 and \mathcal{C}_2 that fall within the physical region, which is the triangle delimited by the lines $x = 0$, $y = 0$ and $y = 1 - x$. The first conic section, \mathcal{C}_1 , is a non-degenerate hyperbola, while the second one, \mathcal{C}_2 , factorizes into the product of the two straight lines:

$$\begin{aligned} x &= 0 & \mathcal{R}_1 \\ y &= \gamma^{-1}x + \gamma^{-1}(r\beta - 1) & \mathcal{R}_2. \end{aligned} \quad (4)$$

Only one of the two intersections of \mathcal{R}_1 with \mathcal{C}_1 , the one at $P_1 = (0, 1)$, falls within the physical region; it is a stationary solution for any set of parameters values and for any r . P_1 corresponds to the trivial absorbing state of unanimous consensus on opinion A . The collective behavior of the system for any given set of parameters is fully determined by the possibility that other non-trivial fixed points (any possible intersections of \mathcal{C}_1 and \mathcal{R}_2) appear within the physical region as the control parameter r is varied. The existence, position and stability of non-trivial fixed points depend on the specific values of the six parameters $(\gamma_1, \gamma_2, \varphi_1, \varphi_2, \beta_1, \beta_4)$ defining the model.

In what follows we classify each model, specified by a set of parameters, according to its collective behavior as the control parameter r is varied (r is a monotonic function of the fraction p of committed agents). The behavior at large r is independent of the parameters values; trivially, in that regime, the full consensus on opinion A is the only stable stationary state. However, in the small r regime, we observe four qualitatively distinct types of response to the variation of the control parameter. We therefore classify all possible specific models, each one specified by a set of parameters, in four classes. The phase diagram for each class is shown in Fig. 1, while the “meta-phase diagram” giving the classification of the models according to their set of parameters is summarized in Fig. 2. We now describe the phenomenology for each one of the four classes, and postpone the detailed analysis of the solutions of Eqs. (3) to the next section.

Class I: Finite Critical Mass (FCM) Models

When $\varphi_1 < 0$, $\gamma_1 \leq 0$, i.e. in models where $A - B$ interactions produce on average an increase in undecided individuals, with no net gain in A nor in B states, the system undergoes a first order transition at a finite value

$r = r_c$ of the external bias [see Fig. 1(a)]. For large enough fraction of committed (large r), P_1 is the only fixed point and the system flows into the absorbing state of total consensus on opinion A , for any initial condition. At $r = r_c$ the system undergoes a saddle-node bifurcation [16]: one double solution appears. As r is decreased below r_c the double solution splits into one stable (the one with larger n_B), and one unstable solutions. In the nontrivial stable fixed point the two opinions A and B coexist in the population, together with a fraction of undecided [see Fig. 1(a)]. We call this state, where disagreement persists, “pluralism”. The initial conditions determine whether consensus is asymptotically reached ($n_A = 1$), or disagreement persists ($n_B > 0$). The value of n_B at the unstable fixed point stays finite in the limit $r \rightarrow 0$, implying that, no matter how small is the fraction of individuals committed to opinion A , a finite “critical mass” [17] of dissenters (supporting opinion B in the initial state) is always necessary to reach the pluralistic state.

Class II: Vanishing Critical Mass (VCM) Models

This class is identified by the range of parameters $\varphi_1 < 0$, $0 < \gamma_1 < \min(\gamma_1^*, -\phi_1)$ [see below for the definition of γ_1^*], and corresponds to models where $A - B$ interactions cause on average a small increase of B states at the expense of A states. As in *Class I* models, lowering r below r_c the system undergoes a first order transition separating a regime ($r > r_c$), where consensus on opinion A is the only stable state from a regime ($r \leq r_c$) where a stable and an unstable additional fixed points appear through a saddle-node bifurcation (see next section). However, in this case, further reducing r , the unstable fixed point collides with the point P_1 (consensus on A) at a finite value r^* ($0 < r^* < r_c$), and then exits the physical region. Correspondingly, the value of n_B at the unstable fixed point vanishes when $r \rightarrow r^*$. This is a transcritical bifurcation [16]: when the two fixed points cross each other, they exchange stability; P_1 becomes unstable, so that below r^* the system, unless started with $n_B \equiv 0$, always flows to the pluralistic state [Fig. 1(b)]. Therefore in this case, the initial presence of even a few dissenters suffices for opinion B to survive. The curve $\gamma_1 = \gamma_1^*$ separating *Class II* and *Class III* depends on all other model parameters. In a certain range of parameters γ_1^* becomes larger than $-\varphi_1$ so that all physical values of γ_1 are always below γ_1^* , and *Class III* disappears [see Fig. 2(c)]. The transcritical bifurcation line is $r = r^* = \gamma_1/(\beta_1 + \beta_4)$, and always lies below the saddle-node bifurcation line.

Class III: Zero Critical Mass (ZCM) Models

When $\varphi_1 < 0$, $\gamma_1 \geq \gamma_1^*$, corresponding to models where $A - B$ interactions give an increase of undecided and a

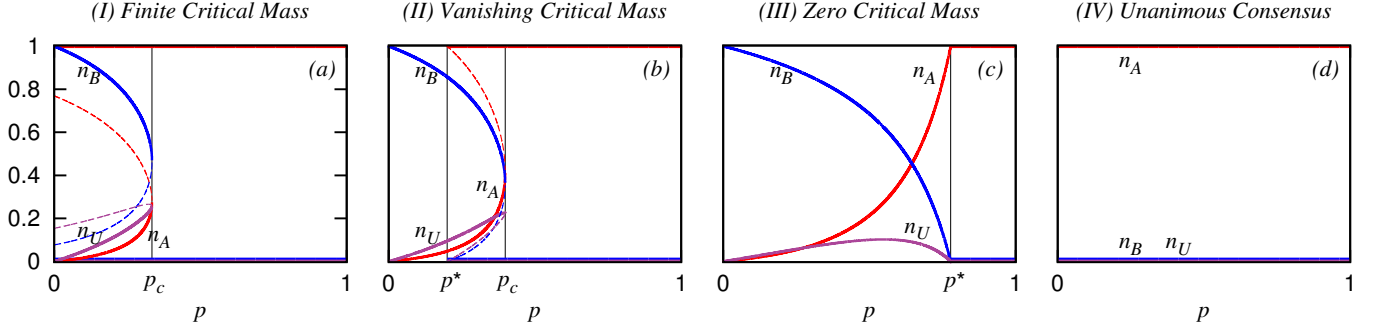


FIG. 1. (color online) Theoretical results for the densities of agents for realizations of each of the four classes of models (a) *Class I* model (FCM) ($\varphi_1 = -0.8$, $\varphi_2 = 0.4$, $\gamma_1 = -0.1$, $\gamma_2 = 0.5$, $\beta_1 = 0.1$, and $\beta_4 = 0.1$), (b) *Class II* model (VCM) ($\varphi_1 = -0.8$, $\varphi_2 = 0.4$, $\gamma_1 = 0.05$, $\gamma_2 = 0.5$, $\beta_1 = 0.1$, and $\beta_4 = 0.1$), (c) *Class III* model (ZCM) ($\varphi_1 = -0.8$, $\varphi_2 = 0.4$, $\gamma_1 = 0.7$, $\gamma_2 = 0.5$, $\beta_1 = 0.1$, and $\beta_4 = 0.1$), (d) *Class IV* model (TC) ($\varphi_1 = 0.5$, $\varphi_2 = 0.4$, $\gamma_1 = -0.5$, $\gamma_2 = 0.5$, $\beta_1 = 0.1$, and $\beta_4 = 0.1$). Solid (dashed) lines represent stable (unstable) lines.

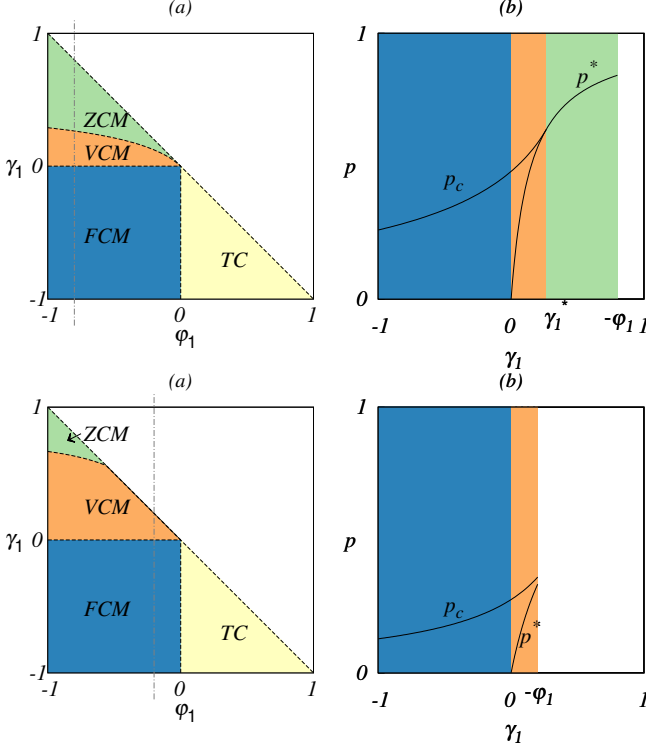


FIG. 2. (color online) (a,c) Phase diagram in the (φ_1, γ_1) plane. The curve γ_1^* separating regions VCM and ZCM depends on φ_2 , γ_2 , β_1 and β_4 . In the upper panel (a) $\varphi_2 = 0.4$, $\gamma_2 = 0.7$, $\beta_1 = .2$, and $\beta_4 = .2$, so that $\varphi_2 > \gamma_2 \beta_4 / (\beta_1 + \beta_4)$. In the lower panel (c) $\varphi_2 = 0.05$, $\gamma_2 = 0.8$, $\beta_1 = .2$, and $\beta_4 = .2$, so that $\varphi_2 < \gamma_2 \beta_4 / (\beta_1 + \beta_4)$. (b,d) Plot of p_c (saddle-node bifurcation curve) and p^* (transcritical bifurcation line) as functions of γ_1 , and for $\varphi_1 = -0.8$ (upper panel, (b)), $\varphi_1 = -0.2$ (lower panel, (d)).

large increase in B at the expense of A , the system undergoes, at $r = r^*$, a continuous transition (transcritical bifurcation) between total consensus on opinion A ($r > r^*$) and pluralism ($r < r^*$), see Fig. 1(c). In this

class of behavior initial conditions do not play any role.

Class IV: Total Consensus (TC) Models

This class corresponds to the region $\varphi_1 \geq 0$, i.e. to models where $A - B$ interactions result in a net increase of individuals holding opinion A . The behavior is trivial: irrespectively of the value of all other parameters, for any initial condition, and no matter how small the density p of committed individuals is, the system always converges to the consensus state (P_1) [see Fig. 1(d)].

V. DERIVATION OF THE PHASE DIAGRAMS

We now show how the rich phenomenology described in the previous section and summarized in Fig. 1 is derived from Eqs. (1). We need to find the solutions of Eqs. (3) and look at how their behavior, as the control parameter r (or equivalently p) varies, depends on the parameters defining the model. Depending on the values of the six parameters ($\gamma_1, \gamma_2, \varphi_1, \varphi_2, \beta_1, \beta_4$) controlling the interactions, different collective behaviors emerge.

In order to find the intersections of \mathcal{C}_1 with \mathcal{R}_1 and \mathcal{R}_2 that correspond to physically meaningful fixed points, it is useful to first identify the position of \mathcal{C}_1 with respect to the physical region. This can be done by locating the intersections of \mathcal{C}_1 with the boundary of the physical region, delimited by the three lines $x = 0$, $y = 0$, $y = 1 - x$. (i) The intersections of \mathcal{C}_1 with $x = 0$ are in $P_1 = (0, 1)$ and $P_2 = (0, -r)$; only the first one belongs to the physical region. (ii) The only intersection of \mathcal{C}_1 with $y = 0$ is in $(\alpha^{-1}, 0)$, which, since $\alpha \leq 1$, is always outside the physical region. (iii) The intersections of \mathcal{C}_1 with $y = 1 - x$ are in $P_1 = (0, 1)$ and $S_1 = (1 - r/\tilde{r}, r/\tilde{r})$, with $\tilde{r} = (\varphi + 1)/(\alpha - 1) = -\varphi_1/\beta_1$; S_1 belongs to the physical region only for $\varphi_1 \leq 0$ and $r \leq \tilde{r}$. Note that whenever S_1 belongs to the physical region (for $\varphi_1 \leq 0$ and $r \leq \tilde{r}$) the upward concavity of \mathcal{C}_1 guarantees that

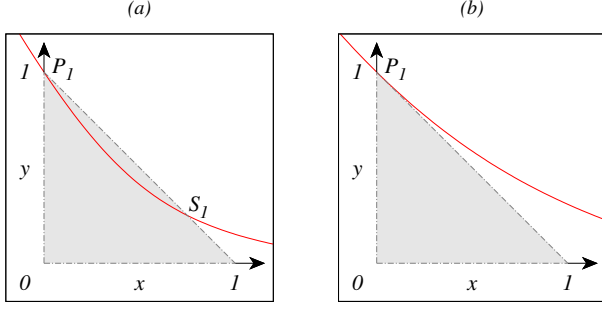


FIG. 3. (color online) Sketch of the qualitative change of the curve \mathcal{C}_1 (solid line) for $\varphi_1 \leq 0$ and $0 < r \leq \tilde{r}$ [panel (a)] and for all other cases [panel (b)]. The grey shaded domain is the physical region.

\mathcal{C}_1 actually enters the physical region. The shape of \mathcal{C}_1 is qualitatively represented in Fig. 3, where (a) corresponds to $\varphi_1 \leq 0$ and $0 < r \leq \tilde{r}$, and (b) to all other cases where \mathcal{C}_1 intersects the physical region in just one point (P_1).

$P_1 = (0, 1)$ always belongs both to \mathcal{C}_1 and to \mathcal{R}_1 (P_1 is in fact the only intersection of \mathcal{R}_1 with the upper branch of \mathcal{C}_1), therefore P_1 is a fixed point for all possible choices of the parameters and for any r . It corresponds to the trivial absorbing state of unanimous consensus on opinion A. For $\varphi_1 \geq 0$ (corresponding to *Class IV* models) \mathcal{C}_1 intersects the physical region only in P_1 [see Fig. 3(b)], thus in this case there cannot be other fixed points besides P_1 . Thus we restrict the following analysis to the non-trivial case $\varphi_1 < 0$: in this case, while P_1 still is the only fixed point for sufficiently large fraction of committed agents [$r \geq \tilde{r}$, see Fig. 3(b)], additional fixed points may arise for $r < \tilde{r}$ from the intersections of \mathcal{R}_2 and \mathcal{C}_1 within the physical region.

$$r = 0$$

To understand how the fixed points move, as the control parameter r is varied, it is useful to first understand what happens in the case $r = 0$, that is the limit of vanishing density of committed agents. For $r = 0$ the line \mathcal{R}_2 always goes through $Q_1 = (1, 0)$, that also belongs to the asymptote $y = 0$ of \mathcal{C}_1 , and is therefore a fixed point. It is clear that, since the slope of \mathcal{R}_2 is γ^{-1} , for $\gamma > -1$ ($\gamma_1 > 0$) Q_1 is the only point of \mathcal{R}_2 that falls within the physical region (see Fig. 4). Let us rewrite \mathcal{C}_1 as

$$(\varphi y - r\alpha)(\varphi y - \varphi^2 x - \varphi + r(\varphi + \alpha)) = r(\varphi + \alpha)(\varphi - r\alpha) \quad (5)$$

that allows to identify the asymptotes. In the limit $r \rightarrow 0$ \mathcal{C}_1 factorizes in the product of its asymptotes, that, in the same limit, are $y = 0$, and $y = \varphi x + 1$. Note that $\varphi_1 < 0$ implies $\varphi < -1$, so that both asymptotes cross the physical region at $r = 0$. When $r = 0$ the intersections of \mathcal{C}_1 with \mathcal{R}_2 are $Q_1 = (1, 0)$ (consensus on opinion B, always within the physical region), and $Q_2 = ((1 + \gamma)/\Gamma, (1 + \varphi)/\Gamma)$, where $\Gamma = 1 - \gamma\varphi$. Q_2 falls within the

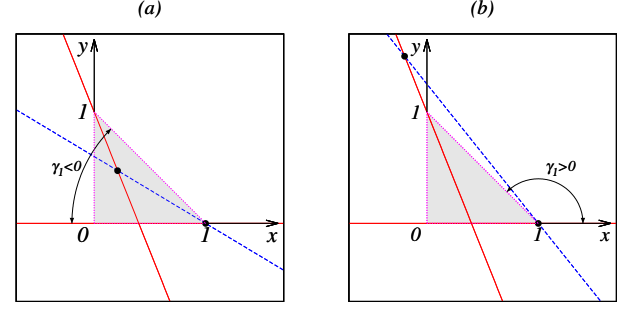


FIG. 4. (color online) Different qualitative behaviors of the stationary solutions for $r = 0$. The red solid line is the curve \mathcal{C}_1 . The blue dashed line is \mathcal{R}_2 . (a) For $\gamma_1 > 0$ the intersection Q_2 of the two curves falls outside the physical region (grey-shaded). (b) For $\gamma_1 < 0$ Q_2 is instead within the physical region. The black curved lines indicate the range of variability of the slope of \mathcal{R}_2 as γ_1 is changed.

physical region as long as $\gamma_1 \leq 0$ (that implies that $1 + \gamma$ and $1 + \varphi$ have the same sign, and therefore Q_2 belongs to the first quadrant), see Fig. 4(a). Q_2 coincides with P_1 for $\gamma_1 = 0$. For $\gamma_1 > 0$ Q_2 is always outside the physical region (to the left of P_1 for $\Gamma < 0$ [see Fig. 4(b)] or to the right of Q_1 for $\Gamma > 0$). For $\gamma > -1$ ($\gamma_1 > 0$) there are overall three fixed points in the physical region: P_1 (unanimous consensus on opinion A) at the intersection of \mathcal{C}_1 with \mathcal{R}_1 ; Q_1 (unanimous consensus on opinion B) at the intersection of \mathcal{C}_1 with \mathcal{R}_2 ; $(0, 0)$, given by the other intersection of \mathcal{C}_1 with \mathcal{R}_1 . For $\gamma < -1$ ($\gamma_1 < 0$) instead, \mathcal{R}_2 always crosses the physical region, and intersects the other asymptote in some Q_2 inside the physical region, giving another fixed point, besides the two corresponding to unanimous consensus, and the one in $(0, 0)$. Note that for $r = 0$ the fixed point in $(0, 0)$ always is unstable, and disappears as soon as $r > 0$.

$$r > 0$$

As r increases, \mathcal{R}_2 always moves leftward (the slope does not depend on r and the intersection with the x axis is $(1 - r\beta, 0)$ always moves leftward as r increases). In the meantime, the upper branch of \mathcal{C}_1 moves rightward [18]. Therefore, the intersection Q_1 always moves leftward. We must now distinguish two cases:

- 1) \mathcal{C}_1 and \mathcal{R}_2 having one intersection on each branch: this occurs for $\Gamma = 1 - \gamma\varphi > 0$. In this case the situation is simple: Q_1 moves leftward as r increases and for some value $r = r^*$ reaches P_1 and then leaves the physical region. The other intersection Q_2 is always on the unphysical branch of \mathcal{C}_1 (the intersection between \mathcal{R}_2 and the line $y = 1 - x$, located in $(1 - r/\tilde{r}, r/\tilde{r})$, with $\tilde{r} = \gamma_1/(\beta_1 + \beta_4)$ is always to the left of S_1 , since $\tilde{r} \leq \tilde{r}$ follows from the physical constraint $\gamma_1 + \varphi_1 < 1$, therefore Q_1 can only exit the physical region crossing P_1).
- 2) \mathcal{C}_1 and \mathcal{R}_2 having two intersections on the same branch: this occurs for $\Gamma = 1 - \gamma\varphi < 0$ (and small enough

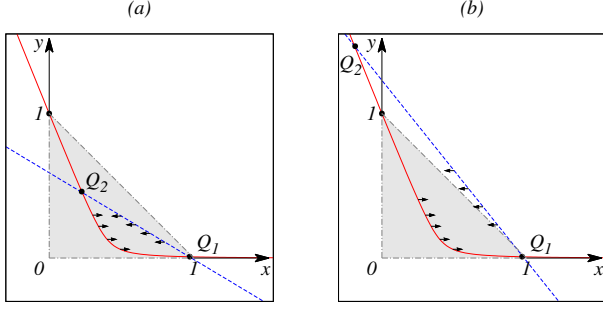


FIG. 5. (color online) Sketch of the motion of the curves \mathcal{C}_1 (red solid curve) and \mathcal{R}_2 (blue dashed line) when r is increased. (a) The intersection point Q_1 moves leftward, while Q_2 moves rightward. For $r = r_c$ they collide and annihilate. (b) The intersection point Q_1 moves leftward, while Q_2 moves rightward. They may collide and annihilate either inside or outside the physical region.

r) (see Fig. 5). In this case, \mathcal{R}_2 always has two intersections with \mathcal{C}_1 for $r = 0$: Q_1 at $(1, 0)$ that moves to the left as r increases, and Q_2 , that is within the physical region for $\gamma_1 < 0$, and outside the physical region (to the left of P_1) when $\gamma_1 > 0$. In both cases, as r increases Q_1 moves leftward, and Q_2 moves rightward, and they collide at some point for some $r = r_c$. When $\gamma_1 < 0$ (Q_2 within the physical region for $r = 0$) Q_1 and Q_2 collide for some finite $r = r_c$. For $r > r_c$. In the other case, $\gamma_1 > 0$, again, as r increases Q_1 and Q_2 get closer and closer (Q_1 moving leftward, and Q_2 moving rightward), but two different events may happen depending on the value of γ_1 : Q_1 and Q_2 either collide within the physical region (Q_2 crosses P_1 for some $r = r^*$ entering the physical region, and then Q_2 collides with Q_1 for some $r_c > r^*$), or they collide outside the physical region (Q_1 crosses P_1 for some $r = r^*$, exiting the physical region and then Q_1 collides with Q_2 at some unphysical point to the left of P_1 for some $r_c > r^*$). A special value $\gamma_1 = \gamma_1^*$ separates the two behaviors: γ_1^* is the value such that Q_1 and Q_2 collide exactly in P_1 , on the boundary of the physical region.

Derivation of r^* and γ_1^*

To obtain the critical value r^* , we observe that r^* is such that at least one intersection between \mathcal{C}_1 and \mathcal{R}_2 coincides with P_1 , which requires that \mathcal{R}_2 goes through P_1 , and therefore

$$r^* = \gamma_1 / (\beta_1 + \beta_4). \quad (6)$$

A critical value r_c such that \mathcal{C}_1 and \mathcal{R}_2 have a double intersection on the upper branch always exist under the necessary condition $\Gamma < 0$, however it is physically meaningful only as long as such double intersection falls within the physical region. We have seen that Q_1 and Q_2 certainly collide within the physical region for $\gamma_1 < 0$, and

for small enough positive γ_1 ($0 < \gamma_1 < \gamma_1^*$). γ_1^* is defined as the value of γ_1 such that Q_1 and Q_2 collide in P_1 , on the boundary of the physical region. Imposing this condition we find that the value of r_c at γ_1^* is $r_c = r^* = \gamma_1^* / (\beta_1 + \beta_4)$, and that γ_1^* is the positive solution of

$$\alpha(\gamma_1^* / \gamma_2)^2 + \gamma_1^* / \gamma_2 (\varphi\beta + \alpha - 1) + \beta(1 + \varphi) = 0. \quad (7)$$

One further piece is missing: while for any given φ_1 a positive value of γ_1^* always exists, it could result to be unphysical in certain ranges of the other parameters φ_2 , γ_2 , β_1 , and β_4 . Indeed, besides being positive, physically meaningful values of γ_1 have to satisfy the condition $\gamma_1 + \varphi_1 \leq 1$. Imposing that $\gamma_1^* < -\varphi_1$ and after some lengthy but trivial calculation we find the condition $\varphi_2 \geq \gamma_2\beta_4 / (\beta_1 + \beta_4)$. In the parameter region $\varphi_2 < \gamma_2\beta_4 / (\beta_1 + \beta_4)$, γ_1^* becomes larger than the upper bound $-\varphi_1$, therefore physically accessible values of γ_1 are always smaller than γ_1^* , and the corresponding models are in *Class III* for any physical positive γ_1 ($0 < \gamma_1 < -\varphi_1$).

Wrap-up: Building the phase diagram from the fixed points

We now have all the elements to summarize the different classes of collective behavior, that depend crucially on the parameters φ_1 and γ_1 . For large r the trivial, stable absorbing state of unanimous consensus on opinion A (P_1) is the only fixed point for all possible sets of parameter values. When $\varphi_1 > 0$ the state of unanimous consensus on opinion A is the only fixed point for any r (*TC* models). When $\varphi_1 \leq 0$, we can distinguish three qualitative different behaviours according to the value of γ_1 : (i) For $\varphi_1 < 0$, $\gamma_1 < 0$ (*FCM* models), as r becomes smaller than a critical value r_c two additional fixed points are formed by means of a saddle-node bifurcation (discontinuous transition). The two fixed points remain physical for any value of r down to $r = 0$. (ii) For $\varphi_1 < 0$, $0 \leq \gamma_1 \leq \min(\gamma_1^*, -\varphi_1)$ (*VCM* models) there is a discontinuous transition at $r = r_c$. When r is further reduced one of the fixed points collides with P_1 for $r = r^*$ and then exits the physical region. At $r = r^*$ the fixed point P_1 becomes unstable and the non-trivial fixed point Q_2 remains the only stable stationary state, unavoidably reached by the dynamics. (iii) For $\varphi_1 < 0$, $\min(\gamma_1^*, -\varphi_1) \leq \gamma_1 \leq -\varphi_1$ (*ZCM* models), at $r = r^*$ the point Q_1 enters the physical region, thus becoming stable, while P_1 turns unstable. A continuous transition occurs between unanimous consensus on A (for $r > r^*$) and a pluralistic state with $\tilde{n}_B > 0$ (for $r < r^*$).

For $\varphi_2 < \gamma_2\beta_4 / (\beta_1 + \beta_4)$ γ_1^* is always below $-\varphi_1$ corresponding to the phase diagram in Fig 2(a). When $\varphi_2 \geq \gamma_2\beta_4 / (\beta_1 + \beta_4)$ instead γ_1^* crosses $-\varphi_1$ at $\varphi_1 = \varphi_1^* = -(\gamma_2\beta_4 - \varphi_2(\beta_1 + \beta_4)) / (\varphi_2 + \beta_4)$. In this case, for $\varphi_1 \geq \varphi_1^*$ there is no continuous transition, and the phase diagram is as in Fig 2(c).

VI. DISCUSSION AND CONCLUSIONS

In this paper we have introduced and solved analytically (in mean-field) a very general model for opinions dynamics in the presence of committed agents. We have shown that when opinions spread in a population where a given fraction of individuals have unshakable opinions, four qualitatively distinct kinds of collective dynamics emerge, depending on the specific rules governing the social interactions. We have categorized this very extensive set of opinion dynamics models accordingly within mean-field. To keep the interaction rules very general the models considered depend on a large number of parameters. A high density of zealots trivially drives the system towards total consensus, irrespectively of all the parameters ruling the interactions. The qualitative behavior of the stationary states at smaller densities depends essentially only on the pair (γ_1, φ_1) , which encode information about interactions among individuals with opposite opinions. The other parameters have a role in locating the tricritical line, the saddle-point bifurcation line and the transcritical bifurcation line. In some parameters regions disagreement persists, while in others consensus is achieved. The final state could depend or not on initial conditions, and the behavior at low densities can change smoothly or abruptly to that at high p . In one parameters region consensus is always achieved also at low densities. Our results allow to precisely predict when these different behaviors occur.

Note that, although the parameters regulating the interactions with undecided agents have a marginal role in determining the collective behavior, the presence itself of the U state is crucial for a nontrivial behavior [19, 20].

Above some critical density of zealots, the system always reaches asymptotically a state of unanimous consensus. The stability of the consensus obtained upon fluctuations of the zealots density depends on the specific model. For models the FCM class consensus is very stable: once it is reached, it is maintained when the density of zealots is diminished, even down to zero. In models in the ZCM class dissenters nucleate as soon as the density of zealots fluctuates below the threshold. Models in the VCM class shows an interesting hysteretic behavior: Once consensus is reached with a sufficiently high zealots density, it is maintained when the density goes below threshold, however, large fluctuations below p_c make the system unstable, and at p^* any infinitesimal perturbation causes an abrupt transition to a state with a finite density of B states.

We finally note a counterintuitive behavior that emerges in certain parameters ranges, where B states survive, and may even become the majority, although both the effect of the zealots and the interaction rules are biased against them [21]. The role of the neutral state is crucial in producing such behavior: Although the peer interaction rules favor the A state, the rate at which they occur allows the B state to be favored on average [20].

For every single choice of the parameters our analy-

sis identifies the value of the critical density of zealots ($\bar{p} = \max(p_c, p^*)$), with $p_c = r_c/(1 + r_c)$ and $p^* = r^*/(1 + r^*)$ above which diverse opinions cannot survive. This critical fraction of committed individuals needed to achieve consensus has an upper bound in $\tilde{p} = \tilde{r}/(1 + \tilde{r}) = 1/(1 - \beta_1/\varphi_1)$. In order for this bound to be non-trivial it has to be $\varphi_1 < 0$ (indeed for $\varphi_1 > 0$ the system always reaches the consensus state). The bound \tilde{p} only depends on the ratio β_1/φ_1 that measures the relative net gain (or loss) of individuals in an A state after an $A - B$ interaction when A is committed with respect to the case when A is uncommitted. When the average net gain in $A - B$ interactions with committed A is larger then the average net loss in A due to $A - B$ interactions with uncommitted A , i.e. for $\beta_1 > -\varphi_1$ the bound ensures $\tilde{p} < 1/2$, meaning that a committed minority suffices to drive the system to consensus.

In our analysis, we have considered generic rules for the peer interactions, but we assumed symmetric roles for the two interacting partners. However, we point out that our analysis holds more generally, including cases where the interaction partners have distinct roles (e.g., speaker-listener), as often considered in the literature. This can easily be shown along the same lines followed in Ref. [15], where we have shown that model allowing for asymmetric roles is always equivalent at the mean-field level to its symmetrized version, the outcome of an interaction in the symmetrized model being defined as the average result of two asymmetric interactions with exchanged roles.

Finally, we remind that our results describe stationary properties of the general model under investigation. The time scales needed to reach the steady state may be strongly dependent on the different parameters in Eqs. (2). Moreover, it should be noted that, since only A zealots are present, consensus on the A opinion is the only possible stationary state in a system of finite size. All other mean-field stationary solutions may appear only as quasi-stationary steady-states.

The present investigation allows to recover, in a unified framework, various types of behavior previously uncovered in several other modeling attempts, thus allowing to understand the physical ingredient behind the single observations. In particular some of the dynamics considered in Ref. [8], are special cases of our general model. For example, the “basic model” of Ref. [8] corresponds to $\gamma_1 = \varphi_1 = -1/2$, $\gamma_3 = \varphi_2 = 1/2$, $\beta_1 = 0$, $\beta_4 = 1/2$. From our analytical treatment, it is immediate to conclude that such a model exhibits a discontinuous transition as p is decreased from unanimous consensus to coexistence of individuals supporting both opinions.

An interesting direction for future work would be to test our prediction outside mean-field on both real and synthetic networks. We expect that, unlike the case where the bias in favor of one opinion comes from a broadcast media source [15], the results in the presence of non-trivial topologies could depend on the location of the zealots, and deviate from mean-field.

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 - [18] For a given ordinate $y = \bar{y}$, $x = (1 - \bar{y})(\bar{y} + r)/(r\alpha - \varphi\bar{y})$, so that $\partial_r x = -(\varphi + \alpha)\bar{y}(1 - \bar{y})/(r\alpha - \varphi\bar{y})^2$, which is positive for \bar{y} in the physical region, being $-(\varphi + \alpha) = (\beta_1 - \varphi_1)/\varphi_2 > 0$.
 - [19] Eliminating the possibility for an individual to be undecided leads to a quite trivial behavior: either the system converge to total consensus on opinion A (when asymmetric interactions favors A), or it exhibit a continuous transition between consensus and pluralism (when asymmetric interactions favors B).
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 - [21] Examples are systems in the FCM class that have $\gamma_1 < \varphi_1$ and $\varphi_2 > \gamma_2$ (the first condition means that more B than A states are lost in $A - B$ interactions; the second that individuals in A state are more successful than those in B state in convincing undecided individuals). In this case, both the presence of zealots and the rules of peer interactions favor A , yet for small densities of zealots and suitable initial conditions the system reaches a stationary state with finite density of dissenters.